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Anomalies from the point of view of G-theory

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Abstract. *The connection between group actions and anomalies in gauge theories is studied within the framework of a recently formulated G-theory. This is a systematic approach to deal with a theory with symmetries and to obtain the reduced effective theory out of it, taking into account the stratification which arises naturally by the action of a symmetry group G. Here the extension of some aspects of the G-theory to the infinite dimensional case, appropriate for the discussion of anomalies, has been done. This concept is then applied to a simple model which corresponds to the Aharonov-Bohm effect. The structure of the occurring infinite dimensional objects (manifolds) is analyzed in great detail. The evaluation of the parity anomaly is given explicitly.*

1. INTRODUCTION

There has never been a doubt about the fundamental role symmetries are playing in investigations concerning the interactions of elementary particles. And still, the geometrical way symmetries are incorporated into the modern gauge theories and the topological effects they lead to, were a real surprise to everyone and remain till now an astonishing fact.

The chiral anomalies are an especially suitable example of such topological effects symmetries are producing. Although they were found in special one-loop Feynman diagrams [1], it later became more and more clear that these are essentially topological effects [2] which have their source in the underlying gauge symmetry and which can appear whenever chiral fermions are present, by the process of quantization of a gauge theory. These topological effects refer often to infinite dimensional manifolds, as e.g.

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they are the space of gauge fields $\mathcal{A}$ (connections) and the gauge group $\mathcal{G}$ (the group of all gauge transformations).

We have obviously to discuss the action of a group (finite or infinite dimensional) on spaces which are not finite-dimensional [3,4]. It is remarkable that such spaces ($\mathcal{G}$ or the space of all fields) possess in a natural way also geometrical structure [3]. The physical significance of the above structures is not at all clear till now and may hide important informations about the nature of the interactions of the elementary particles.

The purpose of this paper is to contribute to the understanding of the anomalies as topological effects induced by symmetries. We shall achieve this by extending some aspects of a recently formulated G-theory [5] to the infinite dimensional case (section 2). G-theory [5] is a systematic approach to deal with a theory which is defined on a finite dimensional manifold U possessing a symmetry given by a group G. This includes two essential steps [5]: we first have to consider the stratification, that means U is a disjoined union of fibre bundles, which arises naturally by the group action on a finite dimensional manifold U. The second step is to obtain the effective theory, by the process of geometric reduction, which corresponds to some space M which can be lower dimensional than the space U we have started with, as, for instance, in the Kaluza-Klein case. This can be achieved by excluding the degrees of freedom which depend on the symmetry. In the present paper we want to extend the above formalism to the infinite dimensional case. Here the space we start with, which corresponds to U, is the space of gauge fields $\mathcal{A}$. The relevant symmetry group corresponding to G is the group of gauge transformations $\mathcal{G}$ which acts on $\mathcal{A}$ in a nontrivial way. So we obtain out of $\mathcal{A}$ (which is an effective space), if we want to exclude the gauge degrees of freedom by performing the geometric reduction, the space of the gauge orbits $\mathcal{A}/\mathcal{G}$. This space has a nontrivial topological structure. The effective theory defined on it has many interesting aspects in connection with the question of anomalies. The G-theory approach proves to be especially useful to describe anomalies.

In particular, we shall show that *anomalies are obstructions* of the quantization procedure which we meet in the process of *reduction within the G-theory framework* (sect. 3).

In addition, we shall demonstrate explicitly the structure of the above mentioned infinite dimensional spaces ($\mathcal{A}$ and $\mathcal{G}$) in a particular simple model which is related to the Aharonov-Bohm effect (sect. 4). In this special model we shall also discuss the question of anomalies from the point of view of the G-theory [5]. We shall show explicitly the non-existence of the usual gauge anomalies, as expected, and we shall show the existence and the evaluation of the parity anomaly (sect. 5).

The main purpose of the sections 4 and 5 is to illustrate in an example chosen as simple as possible some of the main features of the G-theory approach. We have unavoidably to deal with infinite dimensional spaces. In our opinion, considerations concerning infinite dimensional spaces will enter more and more into the phenomenological investigations in future. In order to facilitate the insight into this field for more phenomenologically oriented scientists, we have been quite explicit in the treatment of the above mentioned sections 4 and 5 which can be read parallely to if not completely independently by of the preceding sections.

2. G-THEORY CONCEPT AND GAUGE THEORY

2.1. The gauge theory

An anomaly is connected with an unexpected behaviour when fermions are coupled to gauge bosons. We therefore consider gauge fields corresponding to the compact group G and fermions on a compact Riemannian space M. We are working in the Euclidian regime and we have of course assumed that the space M can possess spinor fields (M must be a spin manifold). The most natural description of gauge fields is given by the connection A on a principal G-bundle$P$ (locally trivial, i.e. $P = M \tilde{\times} G)$. The fermions Ψ are described by a section in a vector bundle W which is a tensor product bundle $W := F \otimes E$ of the spinor vector bundle $F = S \times_{\mathrm{Spin}} \Delta$, with S the spin (dim M) principal bundle (locally $S = M \tilde{\times}\,\mathrm{Spin}$), Δ the corresponding Clifford module and the vector bundle $E = P \times_G V$ with V a representation space of the structure group G. The spin connection is given as a fixed background field. In the following we shall consider, without loss of generality, with given M and G, only *one* fixed principal bundle P or more precisely, one isomorphism class of principal G-bundles. The «full» configuration space of our problem is given by the set of all connections $\mathcal{A}$ on P and all sections in W. We shall call it for good reasons the preconfiguration space $\mathbf{H}$

$$\mathbf{H} = \mathcal{A} \times H = \{(\mathbf{A}, \Psi) | \mathbf{A} \text{ a connection on } P,\ \Psi \text{ a section in } W\}.$$

All function spaces have to be completed to appropriate Sobolev spaces. This is necessary, since we need their Hilbert space structure. The dynamics of our theory is given by the action

$$S(A, \Psi) = S_A + S_\Psi \tag{1}$$

with

$$S_A = \int_M |F|^2 \text{and}\, S_\Psi = \int_M \bar{\Psi}\, \not{D}_A \Psi .$$

F is the curvature form and $\not{D}_A$ is the Dirac operator.

Our system at least will have the gauge invariance described by the gauge group $\mathcal{G}$. That is the vertical automorphisms of P, $\mathcal{G} = \mathrm{Aut}_M P$[6], a subgroup of the $\mathrm{Aut}\, P$, the subgroup of diffeomorphisms on P which *commute* with the action of the group action G on P. We shall in addition assume the existence of a larger symmetry $\tilde{\mathcal{G}}(\mathcal{G} \leq \tilde{\mathcal{G}} \leq \operatorname{Aut} P)$. Since we shall not consider gravitation as a dynamical theory, the metric g_M of our «space-time» M will appear as a background field in the action. So in our theory we shall consider the metric g_M as an absolute element. $\tilde{\mathcal{G}}$ should be the subgroup in $\operatorname{Aut} P$ which *preserves* (after projection) the metric g_M. $\tilde{\mathcal{G}}$ induces on M the isometry group $\hat{G}$ and we have $\hat{G} \cong \tilde{\mathcal{G}}/\mathcal{G}$.

In the following we shall first restrict ourselves to the gauge group $\mathcal{G}$. The action of $\mathcal{G}$ on $\mathbf{H}$ is given by

$$\begin{aligned} &\mathbf{H} \times \mathcal{G} \to \mathbf{H} \\ &((A,\Psi),g) \mapsto (A \cdot g, g^{-1} \cdot \Psi) \end{aligned}$$

with $A \cdot g := g^{-1}Ag + g^{-1}\mathrm{d}g$ and $(g\Psi)(m) := \rho(\tilde{g}(m))\Psi(m), m \in M$ with ρ the representation of G in V, g a section in the G-bundle $M \tilde{\times} G$ with inner group action instead of the right action as in P, corresponding to the element g of the gauge group $\mathcal{G}$. The invariance of the action can now be described by

$$S((A,\Psi)g) = S(A,\Psi).$$

2.2. G-theory

As we have seen, the space $\mathbf{H}$ is a G-space and our theory is assumed to be G-invariant. From the leading principle of the G-theory [5] we may deduce that the above theory can be reduced without any loss of information to an effective theory at the level of the orbit space $\mathbf{H}/\mathcal{G}$. To be more precise, we need some more preparation. Suppose a theory described by the map

$$f : X \to Y$$

where X is a right and Y a left G-space. There is a natural G-action on the space $\operatorname{Map}(X,Y)$

$$G \times \operatorname{Map}(X,Y) \to \operatorname{Map}(X,Y)$$

given by

$$(g,f) \mapsto gf(xg) =: (gf)(x).$$

If we consider those maps $\bar{f}$ where the G-action is trivial (i.e. $g\bar{f} = \bar{f}$), there are *two* possibilities:

(i) Invariance or the equivariance property if

$$\bar{f}(xg) = g^{-1}\bar{f}(x) \Rightarrow g\bar{f} = \bar{f} \qquad \text{(the invariance)}$$

(ii) Strict invariance if G is acting on Y trivially

$$\bar{f}(xg) = \bar{f}(x) \Rightarrow g\bar{f} = \bar{f}.$$

The effective theory is defined after the reduction of the equivariant map

$$\bar{f} : X \to Y$$

to the map

$$\tilde{f} : X/G \to \frac{X \times Y}{G}$$

with $\tilde{f}$ defined by

$$\tilde{f}(m) := [x, \bar{f}(x)] \quad \text{with} \quad m = \pi(x)$$

and

$$\pi : X \to X/G.$$

If we restrict ourselves to a fixed stratum in X, there is a one to one correspondence between $\bar{f}$ and $\tilde{f}$. This procedure is called geometrical reduction. In the case of strict invariance (ii), $\bar{f}$ contains no information related to G and $\tilde{f}$ can be viewed as a map $\tilde{f} : X/G \to Y$, too. In the case (i), some information of G is contained in $\bar{f}$ which after reducing will appear in the orbit space $\frac{X \times Y}{G}$. Applying this idea to gauge theories, we can reduce the full preconfiguration space $\mathbf{H}$ to the space of gauge inequivalent configurations $\mathcal{H} = \mathbf{H}/\mathcal{G}$. For this reason we call $\mathbf{H}$ the preconfiguration space. Since gauge equivalent field configurations are indistinguishable, we expect no physical effects of the symmetry in the reduced theory. Since the action $S(A, \Psi)$ is gauge invariant, this is exactly the result we obtain when we apply the G-theory concept to the case (ii). In classical physics, this can be done without further problems. In quantum physics, the situation may change drastically and we may meet the anomalies as obstructions to the quantization in the process of applying the G-theory concept. The reduction of the preconfiguration space $\mathbf{H}$ leads to the proper configuration space $\mathcal{H} = \mathbf{H}/\mathcal{G}$. In quantum physics such a reduction may be inconsistent with the quantization procedure in the sense that this procedure may lead to an «effective» action which is not *anymore strictly invariant.* In this case, we *have a gauge anomaly.*

If our theory is strictly invariant on a larger group $\tilde{\mathcal{G}}(\mathcal{G} \leq \tilde{\mathcal{G}})$ and if the first step leads to $\mathcal{H} = \mathbf{H}/\mathcal{G}$ in a consistent way with quantization (no gauge anomalies), then we can go to the second step and try to reduce $\mathcal{H}$ further by «dividing» with the group $\hat{G}$. If the second step is not consistent with quantization, then we have $\hat{G}$-anomaly. As we consider the reduction formalism as particularly important for the understanding of anomalies, we shall proceed here, similarly to the finite dimensional case [5], by generalizing it in order to obtain the effective theory and the structure of the corresponding orbit space $\mathcal{H}$.

Our aim is to reduce the preconfiguration space $\mathbf{H}$ by the action of the gauge group $\mathcal{G}$. As a first step, it is useful to parametrize the elements of H by the gauge inequivalent gauge fields. So we have the diagram

$$\begin{array}{ccc} \mathbf{H} = \mathcal{A} \times H & \xrightarrow{Q} & \mathcal{H} \\ \big\downarrow (\pi_{\mathcal{A}},0) & \swarrow & \\ \mathcal{M} & & \pi_{\mathcal{H}} \end{array}$$

where $Q : \mathbf{H} \to \mathcal{H}$ is a projection given by

$$(A,\Psi) \mapsto [A,\Psi] \in \mathcal{H}$$

and

$$\begin{aligned} \pi_{\mathcal{A}} : \mathcal{A} &\to \mathcal{M} := \mathcal{A}/\mathcal{G} \\ A &\mapsto [A]. \end{aligned}$$

The projection $\pi_{\mathcal{H}} : \mathcal{H} \to \mathcal{M}$ is defined by

$$\pi_{\mathcal{H}} \circ Q = \pi_{\mathcal{A}}$$

So the set $\pi_{\mathcal{H}}^{-1}([A])$ consists of those elements in $\mathcal{H}$ which are parametrized by all gauge equivalent connections $[A] \in \mathcal{M}$. In general, $\mathcal{M}$ is a stratified set. The stratification of $\mathcal{M}$ is given by the occurring orbit types of $\mathcal{G}$ in $\mathcal{A}$. These orbit types are classified by the occurring stability groups. So we have, similarly to the finite dimensional case, [5,7] the following result [8]

$$\begin{array}{ccc} \mathcal{A} \cong \bigcup\limits_{i\in I} & & \mathcal{A}_i \\ \big\downarrow \pi_{\mathcal{A}} & & \big\downarrow \pi_i \\ \mathcal{M} \cong \bigcup\limits_{i\in I} & & \mathcal{M}_i \end{array}$$

where I is a partial ordered indexing set and

$$\mathcal{G}/J_i \to \mathcal{A}_i \to \mathcal{M}_i$$

is a smooth fibre bundle with fibre $\mathcal{G}/J_i, J_i < G$ is the orbit type of $\mathcal{A}_i$, that means the stability groups in $\mathcal{A}_i$ are conjugate to J_i. The question about the structure of this fibration arises naturally

$$\pi_{\mathcal{H}} : \mathcal{H} \to \mathcal{M}$$

The orbit bundle $\mathcal{A}_i$ can be regarded as the associated bundle

$$\mathcal{A}_i = \mathcal{A}^{J_i} \times_{N_i} \mathcal{G}/J_i$$

where $\mathcal{A}^{J_i} = \{A \in \mathcal{A}_i | \text{ stability group of } A \text{ is } J_i\}$ and $N_i = N(J_i)/J_i, N(J_i)$ is the normalizer of J_i in $\mathcal{G}$. Starting with $\mathbf{H} = \mathcal{A} \times H$, we have

$$\mathbf{H} = \bigcup_{i \in I} (\mathcal{A}^{J_i} \times_{N_i} \mathcal{G}/J_i) \times_{\mathcal{G}} H.$$

«Dividing» $\mathcal{G}$ yields

$$\mathcal{H} = \bigcup_{i \in I} (\mathcal{A}^{J_i} \times_{N_i} \mathcal{G}/J_i) \times_{\mathcal{G}} H$$

since $\mathcal{A}^{J_i}$ is a right N_i-space, $\mathcal{G}/J_i$ a left N_i and a right G-space and H a left G-space, we have

$$\mathcal{H} = \bigcup_{i \in I} \mathcal{A}^{J_i} \times_{N_i} (\mathcal{G}/J_i \times_{\mathcal{G}} H)$$

[7] and with the isomorphism

$$(\mathcal{G}/J_i \times_{\mathcal{G}} H) \cong H/J_i$$

given by the maps $J_i h \to [J_i e, h]$ and $[J_i g, h] \to J_i g h$ which are inverse to each other, we obtain

$$\begin{array}{ccccc} \mathcal{H} & \cong \bigcup_{i\in I} & \mathcal{A}^{J_i} \times_{N_i} & H/J_i \cong \bigcup_{i \in I} & \mathcal{H}_i \\ \downarrow \pi_{\mathcal{H}} & & \downarrow \pi_{\mathcal{H}_i} & & \downarrow \pi_{\mathcal{H}_i} \\ \mathcal{M} & \cong \bigcup_{i \in I} & \mathcal{M}_i \cong & \bigcup_{i \in I} & \mathcal{M}_i \end{array}$$

So $\mathcal{H}$ is the union of fibre bundles $\mathcal{H}_i$

$$H/J_i \to \mathcal{H}_i \to \mathcal{M}_i$$

with the fibre H/J_i and the structure group $N_i = N(J_i)/J_i$. (1) The last step is to divide $\hat{G}$ out of $\mathcal{H}$ if we have an even larger symmetry group $\widetilde{\mathcal{G}}$. This can be done in the same manner as in the first step. Then the reduced function $\widetilde{S} : H/\widetilde{\mathcal{G}} \to \mathbb{R}$ will contain the dynamics of our theory (2). This completes the reduction of our theory on the classical level.

3. ANOMALIES AND G-THEORY

In order to deal with interactions of elementary particles, we have to leave the classical theory. A quantization procedure is necessary and we shall proceed with Feynman path integral formalism. This is not at all a solved problem and it may very well be that the interpretation of anomalies [9] depends crucially on the solution of the problem of quantization. In spite of this,we shall here take the usual path of Feynman path quantization since this seems at present to be physically the most appropriate approach.

We start with the propagator functional

(2) $$Z : \mathcal{A} \times H \to \mathbb{C}$$

given by

$$Z = \exp(S_A + S_\Psi)$$

and we have to perform the functional integration, assuming the existence of the corresponding appropriate measure we need

$$\int \mathcal{D}A\mathcal{D}\bar{\Psi}\mathcal{D}\Psi \quad Z(A,\Psi).$$

The first step is the Berezin integration over the space of fermions. This leads to an effective propagator functional $\bar{Z}(A)$ which is proportional to the determinant det $\not{D}_A$

$$\det \not{D}_A : \mathcal{A} \to \mathbb{C}$$

of the Dirac operator. If we start with chiral fermions, we obtain the chiral determinant. The determinant can be defined only after one has chosen a regularization and we may think e.g. of the ζ-regularization technique.

The point is that the section

$$\bar{Z} : \mathcal{A} \to \mathbb{C}$$

$$\bar{Z}(A) := \exp S_A \cdot \det \not{D}_A$$

may not be *strictly* invariant (ii) (3) under the action of the relevant group (but only

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(1) This has to be done with respect to the Sobolev classes and Sobolev completion.

(2) In general, the orbit space $\mathbf{H}/\widetilde{\mathcal{G}}$ is not a smooth manifold [5, 7].

(3) See the notation in subsection 2.2.

equivariant or invariant (i)) for any possible regularization we may think of. This has first topological consequences for our reduction procedure since we cannot have simply the reduction of $\bar{Z}$ to a function Z' :

$$Z' : \mathcal{A}/\mathcal{G} \to \mathbb{C}.$$

In this case, we obtain from the reduction a *section in a non-trivial* $\mathbb{C}$-line bundle, the determinant bundle Det'. But this topological effect of the reduction has the consequence that we have to *integrate a section in a twisted bundle.* Since this is not possible, *we cannot perform the quantization.* So, as we see, *having lost the strict invariance* after the integration over the fermions, *we cannot proceed with the quantization.* This is what we call the *gauge anomaly.* Stated differently, we may also say that, in this case, because of the quantization, we can not perform the reduction, and the reduction is only possible if we have the strict invariance property.

Since complex line bundles are classified by the first Chern class $c_1 \in H^2(\mathcal{M}, \mathbb{Z})$, gauge anomalies are related to the topology of $\mathcal{M} = \mathcal{A}/\mathcal{G}$. Since $\mathcal{A}$ is topologically trivial, all nontrivial effects are induced by $\mathcal{G}$ and so have their origin in the symmetry. The situation may be more complicated than described above since in general the space $\mathcal{A}/\mathcal{G}$ may not be a smooth manifold. In this case, the stratification, as described in subsection 2.2., has to be considered in more detail.

Our procedure can be simplified by regarding a subgroup of $\mathcal{G}$ which is acting freely on $\mathcal{A}$. This is the pointed gauge group $\mathcal{G}^*$ [3, 8], an infinite dimensional subgroup of $\mathcal{G}$, with the property that it is the identity on the fibre over a fixed point $x_0 (x_0 \in M)$. If we «divide out» the $\mathcal{G}^*$, only a finite dimensional group is left and we have in particular $\mathcal{G}/\mathcal{G}^* \cong G$, the structure group. Since the action of $\mathcal{G}^*$ on $\mathcal{A}$ is free, we may consider $\mathcal{A}$ as a principal bundle

$$\mathcal{G}^* \to \mathcal{A} \to \mathcal{M}^*.$$

So we obtain by «division» with $\mathcal{G}^*$ from the section $\bar{Z}$

$$\begin{aligned} &\bar{Z} : \mathcal{A} \to \mathcal{A} \times \mathbb{C} \\ &A \mapsto (A, \bar{Z}(A)) \end{aligned} \tag{3}$$

in general, the section Z^* in the determinant bundle

$$\begin{aligned} &Z^* : \mathcal{A}/\mathcal{G}^* \to \frac{\mathcal{A} \times \mathbb{C}}{\mathcal{G}^*} := \mathrm{Det}^* \\ &[A] \mapsto [A, \bar{Z}(A)] \end{aligned} \tag{4}$$

If Det^* is trivial, we may further «divide» with G and we obtain

$$Z' : \mathcal{A}/\mathcal{G} \to \frac{\mathcal{M}^* \times \mathbb{C}}{G} := \mathrm{Det}' \tag{5}$$

If Det' is trivial, too, then Z' is a function

$$Z' : \mathcal{M} \to \mathbb{C}$$

and we have no gauge anomalies. This means that the $\bar{Z}$ we started with is strictly invariant under the action of the gauge group $\mathcal{G}$. Therefore, according to subsection 2.2., the reduction procedure leads to a function on the orbit space $\mathcal{M} = \mathcal{A}/\mathcal{G}$. In this case we have no gauge anomalies.

As mentioned, the space $\mathcal{M}$, being an orbit space, is in general not a smooth manifold. $\mathcal{M}$ is a smooth manifold if e.g. the group $\mathcal{G}$ (or G) is acting simply on $\mathcal{A}$. This is the case with the example we shall discuss in the next section, where G is acting trivially on $\mathcal{A}$.

Assuming now the non-existence of the gauge anomaly, we can try to continue the reduction procedure: If Z' is not strictly invariant under the action of $\hat{G}$, then by «dividing» we obtain a section $\hat{Z}$

$$\hat{Z} : \hat{\mathcal{M}} = \mathcal{A}/\hat{\mathcal{G}} \to \frac{\mathcal{M} \times \mathbb{C}}{\hat{G}} := \widehat{\mathrm{Det}} \tag{6}$$

Our theory contains $\hat{G}$-anomalies and hence we stop with reduction at the level of $\mathcal{A}/\mathcal{G}$.

We have seen that starting with a strictly invariant (ii) theory, the quantization procedure can lead to an effective theory which does not possess this property anymore. At this level, we have to stop also the reduction procedure. *The anomalies may therefore be considered as obstructions of the quantization to the reduction procedure.*

The G-theory is a scheme in which it is possible to describe the various kinds of anomalies. What we need in addition is a way to compute the group actions on Det. This can be done with the different kinds of the Atiyah-Singer theorem.

Before coming to our special example, we would like to turn back to the question of smoothness of the orbit space $\mathcal{M}$. If we want to maintain smoothness in the general case, too, we have to consider the stratification and to restrict ourselves to the principal orbit bundle [5, 7] $\bar{\mathcal{A}}$ (4)

$$\bar{\mathcal{G}} \to \bar{\mathcal{A}} \to \bar{\mathcal{M}} = \bar{\mathcal{A}}/\bar{\mathcal{G}}.$$

Its structure group $\bar{\mathcal{G}} = N(J)/J$ is given by the quotient of the stability group J and the normalizer $N(J)$ of J in the gauge group $\mathcal{G}$. $\bar{\mathcal{M}}$ is now a smooth manifold. Using the result of subsection 2.2., we obtain from the restriction of $\bar{Z}$ to $\bar{\mathcal{A}}$:

$$Z' : \bar{\mathcal{A}}/\bar{\mathcal{G}} \to \frac{\bar{\mathcal{A}} \times \mathbb{C}/J}{\bar{\mathcal{G}}}. \tag{7}$$

(4) This corresponds to the irreducible connections on P.

This result corresponds to eq. (4). From this we can proceed to further reduction. An interpretation of the remaining strata with lower dimensions seems not possible at present.

In the next sections, we shall study the spaces $\mathcal{A}, \mathcal{G}, \mathcal{A}/\mathcal{G}, \mathcal{A}/\widetilde{\mathcal{G}}$ in an explicit model. We shall calculate det $\not{D}_A$ and exhibit the $\widetilde{\mathcal{G}}$ action on $\mathbb{C}$. Every step in the calculation will be done explicitly.

4. THE AHARONOV-BOHM CASE

A physical example with a non-trivial (5) topological structure is the Aharonov-Bohm effect [10]. This is simple enough to allow explicit studies of an infinite dimensional space, like the space of gauge fields $\mathcal{A}$ and the gauge group $\mathcal{G}$. We shall therefore be able to demonstrate explicitly most of the structures discussed in the previous sections. We consider this to be an important task, too, since it seems to us that this is the best way to get some feeling about the «abstract» objects we are dealing with. They are nevertheless directly related to the physical data of the Aharonov-Bohm effect at the classical level of the theory. Only in the quantized version of the model, which we shall discuss in the next section, we will not be anymore directly in the physical situation, since we shall not incorporate the time dependence.

In the Aharonov-Bohm effect the space outside of the solenoid is curvature free (the field strength F is zero) and we can have only flat, but not trivial connections. We therefore consider a time-independent $U(1)$ gauge theory over the one-dimensional sphere S^1. So we have a $U(1)$ principal bundle over S^1. As we have fixed the basis manifold S^1 and the structure group $U(1)$, there exists only one such bundle (up to isomorphism), the trivial one (the torus $T = S^1 \times U(1)$)

$$U(1) \to T \to S^1.$$

Futhermore, as mentioned, all possible connections must be flat, since the field strength is a 2-form over S^1, which vanishes by definition.

So in this case our preconfiguration space $\mathbf{H}$ will have one component only and will be of the form

$$\mathbf{H} = \mathcal{A} \times H.$$

$\mathcal{A}$ is the affine space of connections on the torus T and H is the space of functions $\Psi : S^1 \to \mathbb{C}$. We will do our study in two steps. We will first consider $\mathcal{A}$ only, in this section, and then we will add H, in the next section.

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(5) This stems from the fact that in the Aharonov-Bohm effect we have to exclude the solenoid. This leads to our S^1-space which has the non-trivial cohomology given by $H^1(S^1, Z)$.

4.1. The space $\mathcal{A}$ of connections (gauge fields) on T

The connections $\mathcal{A}$ on T are Lie $(U(1)) = i\mathbf{R}$-valued 1-forms which have the property of strict verticality, i.e. for $A \in \mathcal{A}$

$$A < \xi^* > = \xi \tag{8}$$

where ξ^* is the fundamental vector field corresponding to $\xi \cdot \xi \in \mathrm{Lie}(U(1))$.

Since T is a trivial bundle we can project A onto the base space when we have chosen a reference connection. We fix this reference connection to be the trivial connection which is the Maurer Cartan form θ in $U(1)$, pulled back to T via the projection $\nu : S^1 \times U(1) \to U(1)$

$$A_{\mathrm{ref}} = \nu * \theta .$$

All other connections are flat but not trivial. Now we choose a coordinate system of T as $\{x, \chi\}$ (see fig. 1). In this coordinate system a connection on T can be written as

$$A = A(x)\mathrm{d}\,x + i\mathrm{d}\,\chi .$$

In our convention the reference connection A_{ref} will be

$$A_{\mathrm{ref}} = i\mathrm{d}\,\chi \qquad (A_{\mathrm{ref}}(x) = 0) .$$

The space $\mathcal{A}$ of connections is an affine space. We can check that only a convex combination $tA_1 + (1-t)A_2$ of two connections A_1, A_2 is a connection, but not the sum or the difference because the property of strict verticality (8) will not be fulfilled. If we specify a reference connection, it is clear that A is characterized by the 1-form $A(x)\mathrm{d}\,x$ in S^1, and so we have an isomorphism to a vector space $\mathcal{A}_0$

$$\mathcal{A}_0 \cong \Omega^1(S^1, i\mathbf{R}) \cong \Omega^1(S^1)$$

where the subscript denotes the dependence on the reference connection.

In our case, $\mathcal{A}_0$ is further isomorphic to the function space $C^\infty(S^1)$. Since dim $S^1 = 1$, we have

$$\Omega^1(S^1) \cong \Omega^0(S^1) = C^\infty(S^1) . \tag{9}$$

A connection on a principal fibre bundle means in geometrical terms the splitting $T_P P = H_P \otimes V_P$ of the tangent space in a horizontal H_P and a vertical subspace V_P

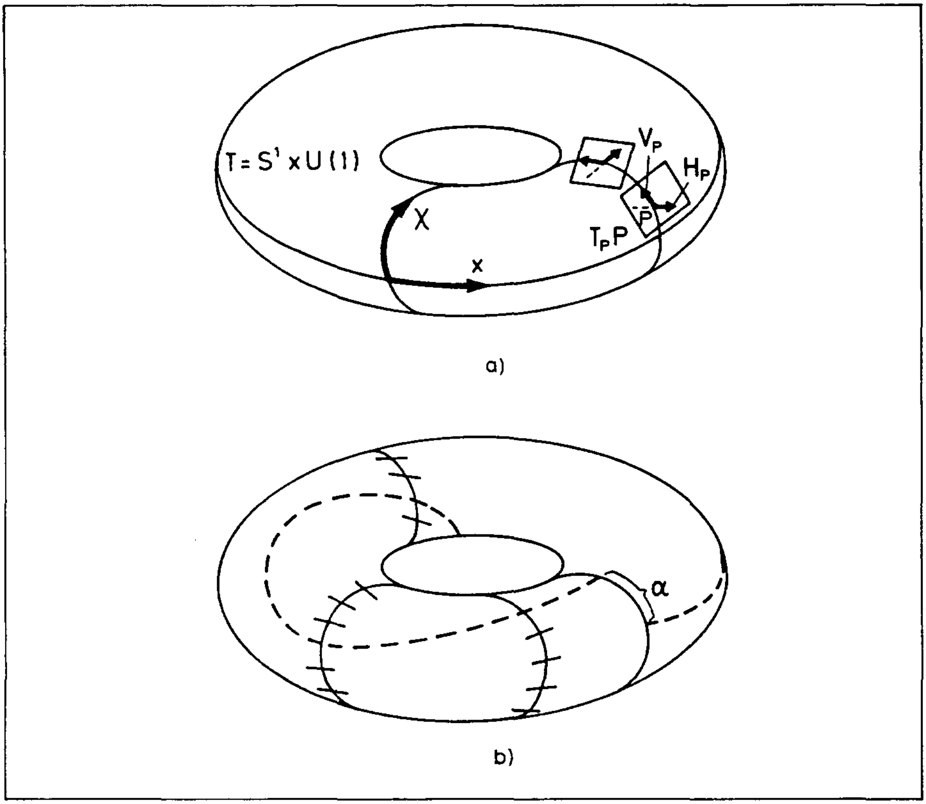


Fig. 1. a) Total space $T \cong S^1 \times U(1)$ of the $U(1)$ principal bundle. (x, χ) are coordinates of base space and fibre . An assignment of a horizontal subspace H_P of tangent space T_PP in every point $p \in P$ corresponds to a connection on P. b) The horizontal curve is tangent to the horizontal spaces. The curve is not A itself, it is a code for A. The defect angle α parametrizes in our case the gauge inequivalent connections.

(see fig. 1). The picture shows the assignment of the horizontal space in every point of T. A horizontal curve will possibly wind around the torus n times and will in general not close by an angle of $\alpha \in [0, 2\pi[$. As we will see later, this angle parametrizes the gauge inequivalent connections on T.

4.2. The gauge group $\mathcal{G}$

The first step has been done. We know the bundle T and the affine space of connections $\mathcal{A}$. Now we want to consider the gauge group $\mathcal{G}$. Before investigating $\mathcal{G}$, we shall give some definitions. The gauge group $\mathcal{G}$ can be viewed as the space of $U(1)$ equivariant maps [11]

$$g_{\text{equ}} : T \to U(1),$$

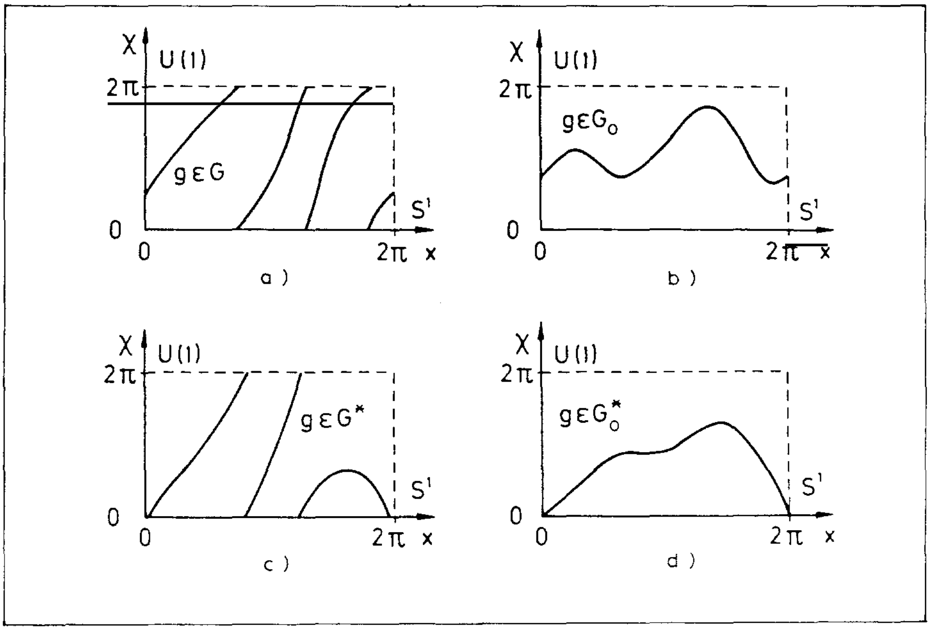


Fig. 2. Graphs of gauge transformations $g \in C^\infty(S^1, U(1))$ a) g element of the full gauge group $\mathcal{G}$. b) g element of the component of the identity $\mathcal{G}_0$ in $\mathcal{G}$. g may not wind around the torus. c) g element of the pointed gauge group $\mathcal{G}^*$. For all g its value is fixed at one point, here $g(0)$ = e.g. may wind around the torus. d) $g \in \mathcal{G}_0^* . g$ has the restrictions of b) and c).

where $(g_1 \cdot g_2)(t) = g_1(t) \cdot g_2(t), t \in T$ gives the group structure.

So we have

$$g_{\text{equ}} : T \to U(1)$$

$$\begin{aligned}(s, u) \to g_{\text{equ}}(s, u) &= g_{\text{equ}}((s, e) \cdot u) \\ &= \text{Int}(u^{-1}) g(s) = u^{-1} g(s) u = g(s)\end{aligned}$$

where $g(s) := g_{\text{equ}}(s, e)$, and where g is actually the map (see Fig. 2a)

$$g : S^1 \to U(1)$$

and $\mathcal{G} \cong C^\infty(S^1, U(1))$. This is an infinite dimensional Lie group [3]. We want to point out that $\mathcal{G}$ is an Abelian group here since $U(1)$ is Abelian. The Lie algebra of $\mathcal{G}$ is given by

$$\text{Lie}(\mathcal{G}) = \{Y | Y : S^1 \to iR\} \cong C^\infty(S^1, i\mathbb{R}).$$

In our case it turns out that we have the vector space isomorphism

$$\mathcal{A}_0 \cong \mathrm{Lie}(\mathcal{G}).$$

We have the exponential map [11]

$$\mathrm{Exp} : \mathrm{Lie}(\mathcal{G}) \to \mathcal{G}$$

defined by

$$\mathrm{Exp}(Y)(s) = \exp Y(s)$$

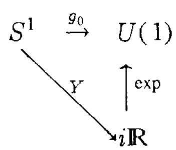
where $s \in S^1$, and with

$$\exp : \mathrm{Lie}(U(1)) \to U(1).$$

Every $g_0 \in \mathcal{G}_0$, which are maps $S^1 \to U(1) \cong S^1$ that are homotopic to the constant map and hence in the component of the identity in $\mathcal{G}$, can be expressed by $Y \in \mathrm{Lie}(\mathcal{G})$ with the help of Exp

$$g_0 = \mathrm{Exp}\, Y.$$

There it is essential that $g_0 \in \mathcal{G}_0$ is an element of the zero class in $\pi^1(U(1))$. Then for every $Y \in \mathrm{Lie}(\mathcal{G})$ there exists a lift to $g_0 \in G_0$



Every map $g : S^1 \to U(1)$ which is an element of the k-class in $\pi^1(U(1))$ has the canonical form

$$g = g_0 g_k$$

where g_0 is element of the o-class and g_k a specific representative of the k-class in $\pi^1(U(1))$, e.g. $g_k(s) = s^k$. From this follows that $\mathcal{G}$ has $\mathbb{Z}$ components labelled by $k \in \pi^1(U(1))$, and every $g \in \mathcal{G}$ can be written as

$$g(s) = (\exp Y(s))s^k \quad k \in \pi^1(U(1)) = \mathbb{Z},\ Y \in \mathrm{Lie}(\mathcal{G}) \cong C^\infty(S^1).$$

We have the product structure (see Fig. 2a, b)

$$\mathcal{G} = \mathcal{G}_0 \times \mathbb{Z}$$

where $\mathbb{Z}$ labels the maps $g \in \mathcal{G}$ given by

$$g(s) = s^k \quad k \in \mathbb{Z}.$$

We want to examine $\mathcal{G}$ a bit closer. The component of the identity of $C^\infty(S^1, U(1))$ is denoted by $C_0^\infty(S^1, U(1))$. If we look at the constant mapping $g \in C_0^\infty(S^1, U(1))$,

$$g = \operatorname{Exp} r \quad r \in C_0^\infty(S^1, i\mathbb{R})$$

$$r : S^1 \to i\mathbb{R} = U(1)$$

$$s \to ir_0, r_0 \text{ constant},$$

we find, when denoting the set of constant maps $r \in C_0^\infty(S^1, i\mathbb{R})$ by $i\mathbb{R}$, that $\operatorname{Exp} i\mathbb{R} = U(1)$ (the structure group) is a subgroup of the gauge group $\mathcal{G} = C^\infty(S^1, U(1))$.

In general, whether $\mathcal{G}$ is connected or not depends on $H^1(M, \mathbb{Z})$ and the topology of the structure group G. In the Aharonov-Bohm case, $\mathcal{G}$ has $\mathbb{Z}$ components because $H^1(S^1, \mathbb{Z}) = \mathbb{Z}$. We see that in the electromagnetic case the question, whether $\mathcal{G}$ is connected or not, depends on the topology of the base manifold M. This is different from the case of an $SU(2)$ gauge theory on S^4. In this case the gauge group is not connected, too, but the reason for this originates in the topology of $SU(2)$ [12]. Now we can go further and study the full symmetry group $\widetilde{\mathcal{G}}$ (subsect. 2.1.). We first study $\operatorname{Aut} P$, of which $\widetilde{\mathcal{G}}$ will be a subgroup. $\operatorname{Aut} P$ consists of diffeomorphisms φ of P,

$$\varphi : S^1 \times U(1) \to S^1 \times U(1)$$

with the property of the $U(1)$-equivariance

$$\varphi(s, u \cdot g) = \varphi(s, u) \cdot g.$$

When P is a trivial bundle as in our case, $\operatorname{Aut} P$ can be represented [6] as a semidirect product of $\operatorname{Aut}_M P = \mathcal{G}$ and $\operatorname{Diff} M$, i.e. there are homorphisms $\alpha : \mathcal{G} \to \operatorname{Aut} P$ and $\gamma : \operatorname{Diff} M \to P$, so that for every $\widetilde{g} \in \operatorname{Aut} P$ there are $g \in \mathcal{G}$, $\hat{g} \in \operatorname{Diff} M$ with $\widetilde{g} = \alpha(g) \cdot \gamma(\hat{g})$. We write $\operatorname{Aut} P = \mathcal{G} \odot \operatorname{Diff} M$. In our example, we consider the subgroup $\hat{G}$ of $\operatorname{Diff} S^1$ which keeps our metric on S^1 invariant. On S^1 we have a constant metric g_{S^1},

$$g_{S^1} : TS^1 \times TS^1 \to \mathbb{R}$$

$$(X,Y) \mapsto X \cdot Y \text{ scalar product in } TS^1|_x \cong \mathbb{R},$$
$$(X,Y) \in TS^1|_x$$

All isometries of S^1 are given by

(i) space reflection $\quad P : S^1 \to S^1$
$\quad s \to \bar{s} \quad s = e^{ix} \in S^1$
or $\quad x \to -x$

(ii) space translation $\quad T : S^1 \to S^1$
$\quad s \to e^{i\alpha} s \quad \alpha \in [0, 2\pi]$
or $\quad x \to x + \alpha$. Hence $\hat{G} = \mathbb{Z}_2 \times S^1$ and our full symmetry group is $\widetilde{\mathcal{G}} = \mathcal{G} \odot (\mathbb{Z}_2 \times S^1)$.

4.3. The space $\mathcal{M}$ of gauge inequivalent connections on T and its stratification

Our next task will be to determine the quotient space $\mathcal{A}/\mathcal{G} =: \mathcal{M}$ which is the true configuration space of our theory. The space $\mathcal{M}$ is the set of equivalence classes [A] given by

$$A' \sim A \Leftrightarrow A' = A \cdot g = A + g^{-1} \mathrm{d} g.$$

With $A \in \mathcal{A}_0$ and $g \in \mathcal{G} \cong C^\infty(S^1, U(1))$ we can proceed in two steps. Because of $\mathcal{G} = \mathcal{G}_0 \times \mathbb{Z}$ and the commutativity of $\mathcal{G}$, we can have

$$\mathcal{M} = \mathcal{A}/\mathcal{G} = \mathcal{A}/(\mathcal{G}_0 \times \mathbb{Z}) = (\mathcal{A}/\mathcal{G}_0)/\mathbb{Z}.$$

So we first want to determine $\mathcal{A}/\mathcal{G}_0$ which means that we have the restricted equivalence class $[A]_0$.

$$A' \sim_0 A \Leftrightarrow A' = A + g_0^{-1} \mathrm{d} g_0 \text{ with}$$
$$g_0 \in \mathcal{G}_0.$$

This g_0 can be written as

$$g_0 = \operatorname{Exp} Y \text{ with } Y \in C^\infty(S^1, i\mathbb{R})$$

and so

$$A' \sim A \Leftrightarrow A' = A + \mathrm{d} Y. \tag{10}$$

Since S^1 is outside of the solenoid, we have zero field strength $(F = 0)$, so that $\mathrm{d} A = F = 0$. A is a closed 1-form in S^1 and the above relation (10) is similar to the one in the de Rahm cohomology. The elements $[A]_0$ of the restricted class $(\underset{0}{\sim})$ are therefore exactly the elements of the first de Rahm cohomology class $H^1(S^1, i\mathbb{R})$. It is known that $H^1(S^1, i\mathbb{R}) = H^0(S^1, i\mathbb{R})$ as may be clear from (9), and so

$$\mathcal{A}/\mathcal{G}_0 = H^0(S^1, i\mathbb{R}).$$

$H^0(S^1, i\mathbb{R})$ is the space of the constant functions in S^1, so we have $H^0(S^1, \mathbb{R}) \cong \mathbb{R}$ and

$$\mathcal{A}/\mathcal{G}_0 \cong \mathbb{R}.$$

We have one more «division» and we obtain

$$\mathcal{A}/\mathcal{G} = (\mathcal{A}/\mathcal{G}_0)/\mathbb{Z} = \mathbb{R}/\mathbb{Z} = S^1.$$

The same result can be obtained by a calculation. Every $A \in \mathcal{A}_0$ can be viewed as a function $A(s)$ with $A = A(s)\,\mathrm{d}\,x$

$$A(s) : S^1 \to i\mathbb{R}$$

and can be expressed in a Fourier series

$$A(s) = i\sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx)$$
$$s = e^{ix}, x \in [0, 2\pi], a_n, b_n \in \mathbb{R}$$

and every $g \in C^\infty(S^1, U(1))$ can be expressed as

$$g(s) = \exp i\left[\sum_{n=0}^{\infty}(\alpha_n \cos nx + \beta_n \sin nx) + kx\right],$$
$$k \in Z, \alpha_n, \beta_n \in \mathbb{R}.$$

Inserting this in

$$A' = A + g^{-1}\mathrm{d}\,g$$

gives

$$a' = a_0 + k$$

where $a', a_0 \in \mathbb{R}$ and $k \in \mathbb{Z}$ since α_n, β_n can be chosen arbitrarily. So we obtain S^1 as a set of equivalence classes, hence

$$\mathcal{M} \cong S^1 .$$

In [5] the aim of G-theories was to reduce the dimensions of a Kaluza-Klein theory from n to 4. Here, in this explicit example, we have found a drastic reduction of an ∞-dimensional problem to a one-dimensional problem on S^1. This is unfortunately not true in general. The reason why $\mathcal{M}$ turns out to be a manifold in this case, is determined by the stratification which we will discuss next.

The stratification of $\mathcal{A}$ by $\mathcal{G}$ is given by the ordered set of the occurring stability groups J. Every J is conjugate to a closed subgroup of the structure group G [7, 8]. This is completely analogous to the situation in [5]. Since $G = U(1)$, the only possible subgroups are Z, Z_n and $U(1)$. Let us study the stability group J_A. $h \in J_A$ iff:

$$A = A \cdot h.$$

This implies that

$$A = A + h^{-1} \mathrm{d} h$$

which gives the condition on $h \in \mathcal{G}$ to be

$$h^{-1} \mathrm{d} h = 0 .$$

We see immediately that we have only one stratum since the condition is independent of $A \in \mathcal{A}$. With $h(s) = \operatorname{Exp} Y(s) s^k$ and $Y(s)$ in coordinates

$$Y(s) = i a_0 + i \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

we find that

$$h^{-1} \mathrm{d} h = Y'(x) \mathrm{d} x + k \mathrm{d} x = 0 \Rightarrow Y'(x) = -k$$

and

$$\sum_{n=1}^{\infty} (-a_n n \sin nx + b_n n \cos nx) = -k .$$

This implies $a_n = b_n = k = 0 \forall n \neq 0$ but a_0 arbitrary. So we find $k = 0$ and $Y(s) = i a_0 \in \mathbb{R} < C^\infty(S^1, i\mathbb{R}) \Rightarrow$

$$h \in J \Leftrightarrow h \in U(1) < C^\infty(S^1, U(1)) .$$

This gives

$$J_A = U(1) \text{for all} A \in \mathcal{A}$$

So we have found that all elements of $\mathcal{A}$ are fixed points of $U(1)$, the constant gauge tranformation. We can now proceed to determine the orbit bundle structure of $\mathcal{A}$.

What structure does $\mathcal{A}$ have? Similar to the finite dimensional case, $\mathcal{A}$ is a bundle over $\mathcal{A}/\mathcal{G}$ with fibre $\mathcal{G}/J[7,9]$. It is easy to see that

$$\mathcal{G} = \mathcal{G}^* \times U(1)$$

where $\mathcal{G}^* = \{g \in \mathcal{G} | g(1) = 1, 1 \in S^1, 1 \in U(1)\}$, the so-called pointed gauge group, i.e. the gauge transformations which leave the fibre over one point in T fixed (Fig. 2c, d). So

$$\mathcal{G}/J = \mathcal{G}/U(1) = \mathcal{G}^* =: C^\infty_*(S^1, U(1)).$$

$\mathcal{A}$ is a bundle with fibre $\mathcal{G}^*$. $\mathcal{A}$ is a $\mathcal{G}^*$ principal bundle over $\mathcal{M}$ since $\mathcal{G}^*$ acts freely on $\mathcal{A}[3]$. In the case we are discussing $\mathcal{G}^*$ will be denoted as $C^\infty_*(S^1, U(1))$ as well.

4.4. The principal bundle $\mathcal{A}$

We have seen that $\mathcal{A}$ is a principal fibre bundle

$$\mathcal{G}^* \to \mathcal{A} \to \mathcal{M}$$

We can ask whether it is trivial or not. If we suppose that it is trivial, it allows a global section σ

$$\mathcal{M} \xrightarrow{\sigma} \mathcal{A} \xrightarrow{\pi} \mathcal{M}$$

with $\pi \circ \sigma = \text{id}$. The application of the homotopy functor would yieald

$$\pi_*(\mathcal{M}) \xrightarrow{\sigma_*} \pi_*(\mathcal{A}) \xrightarrow{\pi_*} \pi_*(\mathcal{M})$$

with $\pi_* \circ \sigma_* = \text{id}$. But $\mathcal{M} \cong S^1$ and $\mathcal{A}$ is an affine space, so we have

$$\pi_*(S^1) = \left\{ \begin{matrix} \mathbb{Z} & k = 1 \\ 0 & \text{else} \end{matrix} \right\} \text{and} \quad \pi_k(\mathcal{A}) = 0 \forall\, k > 0.$$

For $k = 1$ we would have

$$\mathbb{Z} \xrightarrow{\sigma_*} 0 \xrightarrow{\pi_*} \mathbb{Z} \quad \text{with } \sigma_* \circ \pi_* = \text{id}$$

which is a contradiction. This shows that $\mathcal{A}$ is a twisted bundle and therefore there exists no global section.

$$\sigma : \mathcal{M} \to \mathcal{A}$$

This is known in physics as the Gribov ambiguity [3]. Since $\mathcal{A}$ is contractible and a $\mathcal{G}^*$-bundle, it is universal, following a proposition of [13]. A universal bundle helps to classify the isomorphism classes of all $\mathcal{G}^*$-bundles. $\mathcal{M} = \mathcal{A}/\mathcal{G}^*$ is a classifying space $B\mathcal{G}^*$ of $\mathcal{G}^*$-bundles. On the other hand, $\mathcal{M} \cong S^1$ and is a classifying space BZ of Z-bundles where we have the universal Z-principal bundle

$$\mathbb{Z} \to \mathbb{R} \to S^1 .$$

So we have

$$\begin{array}{ccccc} \mathcal{G}^* \to & \mathcal{A} & & \mathbb{R} & \leftarrow \mathbb{Z} \\ & \downarrow & & \downarrow \exp & \\ & B\mathcal{G}^* & = S^1 = & B\mathbb{Z} & . \end{array}$$

In this example, we are in the lucky situation that we can reduce our infinite dimensional principal bundle to a finite dimensional one: the universal covering bundle $\mathbb{R} \to S^1$ of S^1. The reduction is possible if there is a bundle morphism from the bundle $\mathbb{R}$ to the bundle $\mathcal{A}$ with the following properties: The morphism is the identity on the base space and the mapping between the structure groups is an injection [14]. This is clearly the case. The information we need about $\mathcal{A}$ is already contained in the bundle $\mathbb{Z} \to \mathbb{R} \to S^1$. To get some information about how $\mathcal{A}$ is twisted, we have to study only the $\mathbb{Z}$-bundle $\mathbb{R}$. This bundle is given by the projection

$$\pi : \mathbb{R} \to S^1$$
$$\hat{a} \mapsto \exp 2\pi i \hat{a} .$$

Take the covering $D^{\pm}$ of S^1 with the intersection $D^+ \cap D^- = S^0$, where $S^0 = \mathbb{Z}_2 = = \{-1, 1\}$. (Actually we should take an open covering with open intersections.) Bundle charts $\Phi^{\pm}$ are given by

$$\Phi^{\pm} = (\pi; \varphi^{\pm}) : \exp^{-1}(D^{\pm}) \to D^{\pm} \times \mathbb{Z} .$$

For $\hat{a} = a + k$ with $\hat{a} \in \mathbb{R}, k \in \mathbb{Z}$ is

$$\varphi^+(\hat{a}) = k \quad \text{for} \quad a \in [0, \tfrac{1}{2}]$$

$$\varphi^-(\hat{a}) = k \quad \text{for} \quad a \in [\tfrac{1}{2}, 1]$$

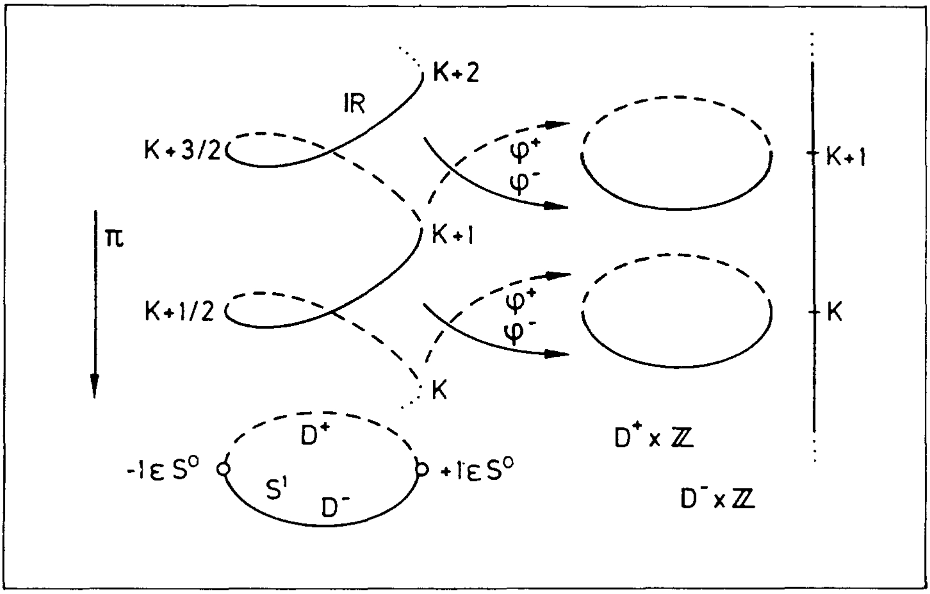


Fig. 3. The universal covering bundle $\mathbb{Z} \to \mathbb{R} \xrightarrow{\pi} S^1$ and a bundle atlas $\Phi = \{\Phi^+, \Phi^-\}$. $\{D^+D^-\}$ is a covering of S with $D^+ \cap D^- = S^0 = \{-1, 1\}$, more precisely if open neighbourhoods are taken. $\Phi^\pm = (\pi, \varphi^\pm) : \pi^{-1}(D^\pm \to D^\pm \times \mathbb{Z}$ are the bundle maps and define a trivialisation. The transition functions $\gamma : D^+ \cap D^- \to \mathbb{Z}$ connect the different trivialisation on the covers D^{pm}.

The transition function γ is given by

$$\gamma : S^0 \to \mathbb{Z}$$

with

$$\gamma(-1) = 0; \ \gamma(1) = 1.$$

For instance $\varphi^+(0) = 0$ and $\varphi^+(1/2) = 0$, so we have for $\hat{a}$ with $\pi(\hat{a}) \in S^0$:

$$\varphi^+(\hat{a}) = \varphi^-(\hat{a}) + \gamma(\pi(\hat{a}))$$

$$\varphi^+(0) = \varphi^-(0) + \gamma(\pi(0)) = -1 + \gamma(1)$$

$$\varphi^+(\frac{1}{2}) = \varphi^-(\frac{1}{2}) + \gamma(\pi(\frac{1}{2})) = 0 + \gamma(-1).$$

This gives a bundle atlas, we have the well known picture, (Fig. 3). How does the bundle atlas look for the bundle $\mathcal{A}$? We have the following charts:

$$\Phi^\pm = (\pi, \varphi^\pm) : \pi^{-1}(D^\pm) \to D^\pm \times \mathcal{G}^*$$

For $A = \hat{a} + \mathrm{d}Y \in \mathcal{A}$ with $\hat{a}$ is a constant valued 1-form with values $\hat{a} = a + k \in \mathbb{R}$, $k \in \mathbb{Z}$ the carts $\phi^{\pm}$ are given by

$$[\varphi^{\pm}(A)](s) = \exp(Y_0(s) - Y_0(1)) \cdot s^{k_\pm}$$

with

$$k_+ = k \quad \text{for } a \in [0, \frac{1}{2}]$$
$$k_- = k \quad \text{for } a \in [\frac{1}{2}, 1]$$

The transition function $S^0 \to \mathcal{G}^*$ is given with the help of $\gamma : S^0 \to \mathbb{Z}$

$$\gamma(1) = 1; \quad \gamma(-1) = 0.$$

So we obtain for A with $\pi(A) \in S^0$

$$[\varphi_+(A)](s) = [\varphi_-(A)](s) \cdot s^{\gamma(\pi(A))}.$$

4.5. The space $\hat{\mathcal{M}} = \mathcal{M}/\hat{G}$

In order to determine the quotient space $\hat{\mathcal{M}} = \mathcal{M}/\hat{G}$, we first have to know the action of the group $\hat{G} = \mathbb{Z}_2 \times S^1$ on the space of connections $\mathcal{A}$. From this information we can then infer the action of $\hat{G}$ on the space of the gauge inequivalent connections $\mathcal{M} = \mathcal{M}/\mathcal{G}$.

The two factors of $\hat{G}$ correspond to the reflections (parity) $\hat{P} = \mathbb{Z}_2$ and to the translations $\tau = S^1$ of the space S^1 see subsection 4.2.). The parity transformation $P \in \hat{P}$ is given by

$$P : s \to s^P = \bar{s} \quad \text{(complex conjugation)} \tag{11}$$

and in coordinates $s = e^{2i\pi x}$ by

$$x \to x^P = -x.$$

This induces for the parity transformation of A (in coordinates $A = A(x)\mathrm{d}x$):

$$P : A \to A^P = A(-x)\mathrm{d}(-x) = -A(-x)\mathrm{d}x$$
$$A^P(x) = -A(-x).$$

A space translation $T \in \tau$ on S^1 is given by

$$s \to e^{2\pi i \alpha} s$$

and in coordinates by

$$x \to x + \alpha.$$

This induces

$$A \to A^T = A(x + \alpha)\mathrm{d}(x + \alpha) = A(x + \alpha)\mathrm{d}x$$

so that

$$A^T(x) = A(x + \alpha).$$

In order to derive (from this) the action of $\hat{G}$ on $\mathcal{M}$ it is useful to consider $A, A^P, A^T \in \mathcal{A}$ (as elements of the principal bundle $\mathcal{A}$) by their corresponding expressions in a given bundle chart. So we have in an obvious notation (compare with previous subsection)

$$\begin{aligned} A(x) &:= a + Y'(x) + k \\ A^P(x) &= -a - Y'(-x) - k \\ A^T(x) &= a + Y'(x + \alpha) + k. \end{aligned}$$

From this, taking the projection on $\mathcal{M}$, we obtain

$$\begin{aligned} \pi(A) &= e^{i2\pi a} \\ \pi(A^P) &= e^{-i2\pi a} \\ \pi(A^T) &= e^{-i2\pi a} \end{aligned}$$

and so obtain for $[A] \in \mathcal{M} \cong S^1$

$$\begin{aligned} &[A] \to [A]^P = \overline{[A]} \\ &[A] \to [A]^T = [A]. \end{aligned} \tag{12}$$

As we see, P acts on $\mathcal{M}$ non-trivially and non-freely. The elements $[A] = 1$ and $[A] = -1$ are fixed points of $\hat{P}$. T acts trivially on $\mathcal{M}$.

The quotient space $\hat{\mathcal{M}} = \mathcal{M}/\hat{G}$ is given by

$$\hat{\mathcal{M}} = \mathcal{M}/\mathbb{Z}_2 \cong S^1/\mathbb{Z}_2.$$

5. ANOMALIES IN THE AHARONOV-BOHM CASE

5.1. The fermions

The next step in our simple model is the introduction of fermions. As discussed in subsect. 2.1., fermions are described by sections in the tensor bundle $F \otimes E$. In the Aharonov-Bohm case we are studying here, both F and E are trivial $\mathbb{C}$-line bundles over S^1, and the same is valid for their tensor product. So the fermions can be described by $\mathbb{C}$-valued functions on S^1 :

$$\Psi : S^1 \to \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} = \mathbb{C}.$$

The space of fermions is essentially given by $C^\infty(S^1, \mathbb{C})$. We should consider the Hilbert space H related to the above smooth functions with the appropriate L^2 or Sobolev completion. However, since the results do not change, we shall do the calculation with the smooth functions and so we shall avoid the complications of functional analysis. The action of $\mathcal{G}$ on H is given by

$$\Psi \xrightarrow{g} g\Psi \quad \text{and} \quad (g\Psi)(s) := \rho(g(s))\Psi(s)$$

where ρ is a representation of $U(1)$ on $\mathbb{C}, s \in S^1$ and $g \in \mathcal{G}$. We shall take for ρ the fundamental representation of $U(1)$. The action of $\hat{G}$ on H is given for the parity transformation P by

$$\Psi \xrightarrow{P} \Psi^P \quad \text{and} \quad \Psi^P(s) := \gamma^0 \Psi(s^P)$$

where $\gamma^0 = i$ the generator of the Clifford algebra and s^P is the parity transformed $s \in S^1$. For the translation T it is given by

$$\Psi \xrightarrow{T} \Psi^T \quad \text{and} \quad \Psi^T(s) := \Psi(s + \alpha)$$

where $s, \alpha \in S^1$. The parity acts non-trivially on the spinor bundle F but trivially on the fibres of E. On the preconfiguration space $\mathbf{H} = \mathcal{A} \times H$, the action $S(A, \Psi)$ is given by

$$S(A,\Psi) = \frac{1}{2}\Big(\int_{S_1} \Psi^* \not{D}_A \Psi + hc\Big).$$

This is so because the curvature part in our problem is zero. In odd dimensions we have $\bar{\Psi} = \Psi^* \cdot \not{D}_A$ is the Dirac operator which is the covariant derivative acting on the section Ψ followed by a Clifford multiplication

$$\not{D}_A := i(\mathrm{d} + A) \quad \text{with} \quad A \in \mathcal{A}_0$$

$\not{D}_A$ is $\mathcal{G}$-equivariant

$$\not{D}_{A\cdot g} = g^{-1}\ \not{D}_A g.$$

This follows immediately from the above definitions and from the action of g on $\mathcal{A}$. As can be verified explicitly, $S(A,\Psi)$ is *strictly* invariant under the group $\widetilde{\mathcal{G}}$. That means that at the classical level the theory can immediately be reduced to $\widetilde{S}$

$$\widetilde{S} : (\mathcal{A} \times H)/\widetilde{\mathcal{G}} \to \mathbb{R}.$$

Our aim is to study the relation between symmetry and anomalies. As we have seen, there are no problems in the classical regime. Problems can occur when we quantize the theory. This will be done next.

In the quantized version of our theory, we have to calculate the Feynman path integral (sect. 3)

$$\int \mathcal{D}\, A\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\, Z(A,\Psi).$$

The integral over the fermionic degrees of freedom leads to the determination of the Dirac operator $\not{D}_A$ and we have for the effective propagator functional, since $S_A = 0$ (see sect. 3)

$$\bar{Z} : \mathcal{A} \to \mathbb{C}$$
$$A \mapsto \bar{Z}(A) = \det \not{D}_A.$$

Note that this $\bar{Z}$ is $\widetilde{\mathcal{G}}$-invariant (equivariant) but not necessarily strictly invariant. In order to calculate $\det\ \not{D}_A$, we first have to determine the spectrum of the Dirac operator and then to choose and perform the regularization. In bundle coordinates of $\mathcal{A}$, the Dirac operator has the form $(A^{\pm} = i(a^{\pm} + Y' + k^{\pm})\mathrm{d}\,x)$

$$\not{D}_{A^{\pm}} = i(\partial_x + i(a^{\pm} + Y' + k^{\pm}))$$

and $Y : S^1 \to \mathbb{R}$ and $k^{\pm} \in \mathbb{Z}$, $a^{\pm} \in [0,1[$. We are now going to study the eigenvalue problem. This is a local problem. With $\hat{a} := a + k$ (hence $\hat{a} \in \mathbb{R}$)

$$\not{D}_A \Psi_n(x) = \lambda_n \Psi_n(x)$$

and

$$i(\partial_x + i(\hat{a} + Y'(x)))\Psi_n(x) = \lambda_n \Psi_n(x)$$
$$\Rightarrow \Psi_n(x) = \Psi_n^0 \exp\Big(-i\int_0^x \mathrm{d}\,x'(\lambda_n + \hat{a} + Y'(x)\Big)$$

Since $\Psi_n(2\pi) = \Psi_n(0)$ there follows

$$\begin{aligned}\Psi_n^0 \exp(0) &= \Psi_n^0 \exp\left(-i\int_0^{2\pi} \mathrm{d}x'(\lambda_n + \hat{a} + Y'(x')\right)\\ &= \Psi_n^0 \exp(-2\pi i(\lambda_n + \hat{a} - i(Y(2\pi) - Y(0)))\\ &= \Psi_n^0 \exp(-2\pi i(\lambda_n + \hat{a}))\end{aligned}$$

since $Y(2\pi) = Y(0)$. This is only possible if $\lambda_n + \hat{a} = n \in \mathbb{Z}$ and we obtain the spectrum (Fig. 4a)

$$\{\lambda_n\} = \{-\hat{a} + n\} = \{-a - k + n\}$$

and the eigenvectors

$$\{\Psi_n(x) = \Psi_n^0 \exp(\text{-inx} - i(Y(x) - Y(0)))|n \in \mathbb{Z}\}.$$

We see that the eigenvalues $\{\lambda_n\}$ are not depending on gauge transformations of the form

$$\mathrm{Exp}(Y) : S^1 \to U(1)$$

i.e. transformations from the component of id of the gauge group. Gauge transformations of the type $g : s \to s^k$ move the whole spectrum by k units. Therefore the whole spectrum is gauge invariant. The gauge invariance of the spectrum follows immediately from the equivariance of $\not{D}_A$

$$\begin{aligned}&\not{D}_{A g}\Psi_n = \lambda_n\Psi_n\\ \Rightarrow\ &\not{D}_A(g\Psi_n) = \lambda_n(g\Psi_n).\end{aligned}$$

The spectrum is invariant under gauge transformations. For $\hat{a} \in \mathbb{Z}$ we have integer valued eigenvalues, otherwise they are real valued. Zero modes can occur, too. The spectrum is asymmetric, a feature of the odd-dimensionality of our space. In even dimensions we would have a symmetric spectrum due to the Z_2-grading of the Clifford algebra (γ_5 operator).

5.2. The regularized determinant

For the calculation of the determinant of the Dirac operator $\not{D}_A$ we use the ζ–function regularization technique [2, 15]. The starting point is the spectrum of the Dirac operator $\not{D}_A$. In our case it is given by

$$\{-\hat{a} + n|n \in \mathbb{Z}\}$$

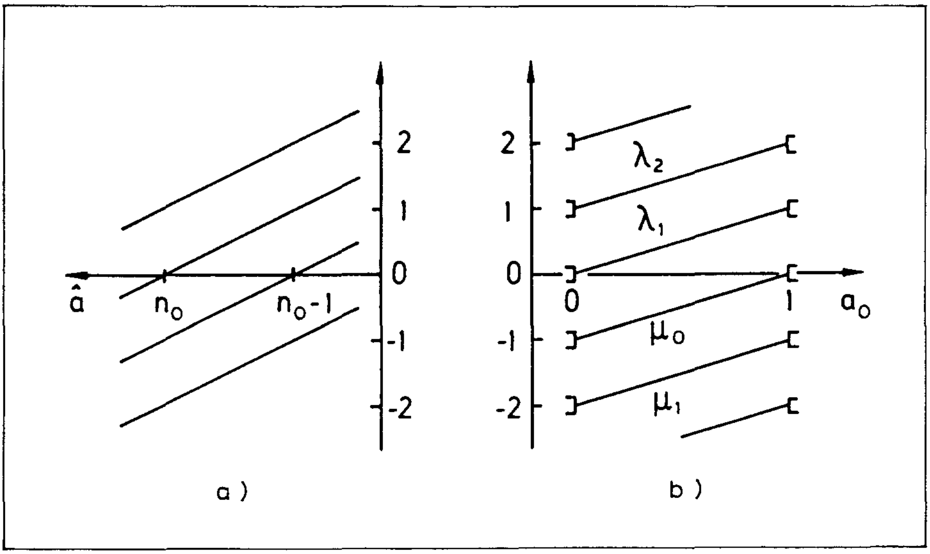


Fig. 4. Spectrum of the Dirac operator on T as a function of a parameter related to the connection on T. a) $\hat{a}$ is the constant part of the connection used in the trivialisation of A. b) a_0 is-$\hat{a}$ modulo integers and is used in the ζ-function regularization of the determinant. a_0 is related to the defect angle α of fig. 1.

with $\hat{a} \in \mathbb{R}$. It contains positive and negative eigenvalues. We therefore use an extension of the usual definition of the determinant via ζ- fucntion, which applies to a self-adjoint elliptic operator with positive symbol, to corresponding operators but with non-positive symbol [15]. For this purpose we split the above spectrum in a positive $\{\lambda_n\}$ and a negative $\{-\mu_n\}$ part (Fig. 4b)

$$\{-\hat{a} + n | n \in \mathbb{Z}\} = \{\lambda_n = a_0 + n | n \in \mathbb{N}\} \\ \cup \{-\mu_n = a_0 - n | n \in \mathbb{N}^+\}$$

where $a_0 = -\hat{a} + n_0$ for $n_0 \in \mathbb{Z}$ such that $a_0 \in]0,1[$. We first exclude the value $a_0 = 0$ since it corresponds to zero modes for which the determinant is not well defined. This determinant is defined by

$$\log \det \not{D}_A := -\frac{d}{ds}\bigg|_{s=0} \left(\sum_{n=0}^{\infty} \lambda_n^{-s} + \sum_{n=1}^{\infty} (-1)^{-s} \mu_n^{-s} \right) \tag{13}$$

where the regularization procedure is in addition implied. It consists in subtracting the pole at $s = 0$. With the choice of a phase $(-)^{-s} = e^{i\pi s}$ and the definitions

$$
\begin{aligned}
\zeta^{(s)}_{|\not{D}_A|} &:= \sum_{n=0}^{\infty} \lambda_n^{-s} + \sum_{n=1}^{\infty} \mu_n^{-s} \\
\eta^{(s)}_{\not{D}_A} &:= \sum_{n=0}^{\infty} \lambda_n^{-s} + \sum_{n=1}^{\infty} \mu_n^{-s}
\end{aligned}
\tag{14}
$$

we obtain

$$
\log \det \not{D}_A = -[\zeta'_{|\not{D}_A|}(0) + \frac{i\pi}{2}\left(\zeta_{|\not{D}_A|}(0) - \eta_{\not{D}_A}(0)\right)]
\tag{15}
$$

ζ' has a pole at $s = 0$ whereas ζ and η are analytic [15]. For the calculation we use the generalized ζ-function of the form

$$
\zeta(s, a_0) := \sum_{n=0}^{\infty} (n + a_0)^{-s} \quad a_0 \neq 0, -1, -2, \ldots
\tag{16}
$$

In our example we have defined a_0 such that

$$a_0 \in]0, 1[.$$

Therefore we have no problems with the definition of the generalized ζ-function. First we calculate $\zeta'_{|\not{D}_A|}(0)$

$$
\begin{aligned}
\zeta'_{|\not{D}_A|}(0) &= \frac{d}{ds}\bigg|_{s=0} \left(\sum_{n=0}^{\infty}(n + a_0)^{-s} + \sum_{n=1}^{\infty}(n + a_0)^{-s}\right) = \\
&= \frac{d}{ds}\bigg|_{s=0} \left(\sum_{n=0}^{\infty}(n + a_0)^{-s} + \sum_{n=0}^{\infty}(n - a_0)^{-s} - (-1)^{-s} a_0^{-s}\right) = \\
&= \frac{d}{ds}\bigg|_{s=0} \left(\zeta(s, a_0) + \zeta(s, -a_0) - e^{i\pi s} e^{-s \log a_0}\right)
\end{aligned}
\tag{17}
$$

We use $\frac{d}{ds}\big|_{s=0} \zeta(s, a_0) = \log \Gamma(a_0) - \frac{1}{2}\log(2\pi)$ [16]. According to the regularization procedure, the pole was substracted (the cut is on the real negative axis). We obtain

$$
\begin{aligned}
\zeta'_{|\not{D}_A|}(0) &= \log \Gamma(a_0) + \log \Gamma(-a_0) - \log(2\pi) - i\pi + \log a_0 \\
&= \log\left(\Gamma(a_0)\Gamma(-a_0)\frac{a_0}{2\pi}\right) - i\pi = \\
&= \log\left(\frac{-\pi}{a_0 \sin(\pi a_0)} \frac{a_0}{2\pi}\right) - i\pi = \\
&= \log(-1) - i\pi - \log(2 \sin(\pi a_0)) = \\
&= -\log(2 \sin(\pi a_0)).
\end{aligned}
\tag{18}
$$

It is easy to see that $\zeta_{|\not{D}_A|}(0) = 0$ where we used $\zeta(0, a_0) = \frac{1}{2} - a_0$ in the expression in brackets in (17). The last step we have to do is to calculate $\eta_{\not{D}_A}(0)$

$$
\begin{aligned}
\eta_{\not{D}_A}(0) &= \left(\sum_{n=0}^{\infty} (n + a_0)^{-s} - \sum_{n=1}^{\infty} (n - a_0)^{-s} \right) \Big|_{s=0} = \\
&= \left(\sum_{n=0}^{\infty} (n + a_0)^{-s} - \sum_{n=0}^{\infty} (n - a_0)^{-s} + (-1)^{-s} a_0^{-s} \right) \Big|_{s=0} = \\
&= (\zeta(s, a_0) - \zeta(s, -a_0) + (-a_0)^{-s})|_{s=0} = \\
&= \frac{1}{2} - a_0 - \frac{1}{2} - a_0 + 1 = \\
&= -2a_0 + 1 .
\end{aligned} \tag{19}
$$

Putting everything together we obtain

$$
\log \det \not{D}_A = \log(2 \sin(\pi a_0)) + \frac{i\pi}{2}(1 - 2a_0)
$$

and we get

$$
\det \not{D}_A = 1 - e^{-2\pi i a_0} .
$$

Because of the asymmetry of the spectrum, our function det is complex valued. We can now substitute our original parameter $a_0 = -\hat{a} + n_0$ and we obtain the final result

$$
\det \not{D}_A = 1 - e^{-2\pi i \hat{a}_0} .
$$

We still have to calculate the determinant for $\hat{a} \in \mathbb{Z}$. We can continue our definition to the case $\hat{a} = 0$ with

$$
\det \not{D}_A \, (\hat{a} \in \mathbb{Z}) = 0 .
$$

This is an analytic continuation. This definition coincides with the requirement that det $\not{D}_A$ is zero if $\not{D}_A$ has zero modes. If $\hat{a} \in \mathbb{Z}$ then $\lambda_n \in \mathbb{Z}$ and $\lambda = 0$ occurs.

5.3. The parity anomaly

In the previous subsection we have obtained an explicit expression for the function $\bar{Z}$. So we are now able to discuss explicitly the question of anomalies. At this point a recapitulation may be useful. We have started with a theory given by the action $S(A, \Psi)$, (1), which was strictly invariant under the symmetry group $\widetilde{\mathcal{G}}$ ($S(A\widetilde{g}, \Psi\widetilde{g}) = S(A, \Psi)$ with $\widetilde{g} \in \widetilde{\mathcal{G}}$, $\widetilde{\mathcal{G}}/\mathcal{G} \cong \hat{G}$. According to the G-theory, we can proceed to the reduction and we can obtain the equivalent action $\widetilde{S}(A, \Psi)$ on the space $(\mathcal{A} \times H)/\widetilde{\mathcal{G}}$. This was the situation at the classical level. Coming to the quantum level, we have to deal with the effective propagator $\bar{Z}(A)$ as given after the integration over the fermionic degress of freedom in the path integral. It may now happen that this $\bar{Z}(A)$ is no longer *strictly* invariant under the action of a subgroup $\hat{G}$.

In this case where the symmetry is broken at the quantum level, we speak of $\hat{G}$–anomaly. It is clear that this is the result of the quantization procedure and we have to stop here with the reduction formalism. It is interesting to note that with the above G-theory point of view, all possible anomalies can be treated in a similar way.

In what follows we shall demonstrate explicitly the above considerations with our model. As we shall see, in our case the $\hat{G}$-anomaly will be the parity anomaly. The discussion and the results in the subsections 5.1. and 5.2. show that there are no gauge anomalies in our model. The function $\bar{Z}$ is strictly $\mathcal{G}$-invariant. This is expected, in one dimension, since the Dirac operator is self-adjoined and the spectrum real and invariant. It is well known that the determinant of a gauge invariant spectrum which results from ζ-function regularization is strictly $\mathcal{G}$-invariant. If we had studied the case of even dimensions and the case of chiral fermions (Weyl fermions), the eigenvalue problem would have been different. We would have an eigenvalue problem formulated with a Laplace-like operator which would be neither self-adjoined nor $\mathcal{G}$-equivariant.

On account of the strict $\mathcal{G}$-invariance of $\bar{Z}$, we can proceed to the first step in the reduction:

$$\begin{array}{rccc} \bar{Z}: & \mathcal{A} & \longrightarrow & \mathcal{A}\times\mathbb{C} \\ & \Big\downarrow{\scriptstyle \pi_{\mathcal{A}}} & & \Big\downarrow \\ Z': & \mathcal{A}/\mathcal{G} & \longrightarrow & \mathcal{A}/\mathcal{G}\times\mathbb{C}. \end{array}$$

Z' is given by $Z'([A]) := \bar{Z}(\pi_{\mathcal{A}}(A))$ and we have with $\mathcal{M} := \mathcal{A}/\mathcal{G} \cong S^1$,

$$\pi_{\mathcal{A}}(A) = [A], \quad [A] =: s$$

and a the constant part of the function $A(x)$

$$Z'([A]) = 1 - e^{2\pi i a} = 1 - [A] = 1 - s.$$

Following the G-theory program, we are prepared for the second step in the reduction. Whether this is possible or not depends on the invariance properties of Z' under the group $\hat{G} = \mathbb{Z}_2 \times S^1$. Since the action of S_1 on $\mathcal{M}$ is trivial (4.5), we have to consider only the parity part $\mathbb{Z}_2$

$$\mathcal{M} \times \mathbb{Z}_2 \to \mathcal{M}$$

$$(s, -1) \to s^P = \bar{s}.$$

The test is whether $Z'([A]^P) = Z'([A])$ is direct. From (11) and (12) we have

$$Z'([A]^P) = Z'(s^P) = Z'(\bar{s}) = 1 - \bar{s}$$

since $Z'([A]^P) \neq Z'([A]), Z'$ is not strictly invariant under the parity transformation (although, as we can immediately see, it is still equivariant). According to our previous discussion we have to stop here with the reduction. The subgroup $\mathbb{Z}_2$ of $\hat{G}$ is no more a symmetry at the quantum level and we have a parity anomaly.

SUMMARY

The purpose of this paper was twofold. Firstly we presented the extension of a recently formulated method (G-theory) to anomalies and secondly the treatment of infinite dimensional objects in a simple and physically quite familiar case. It is a well known fact that anomalies are connected with the action of a symmetry group, the gauge group or some other extension of it, on infinite dimensional objects, as e.g. the space of gauge potential and the space of fermionic fields are. They indicate the breaking of the existent symmetry at the quantum level. The application of the G-theory concept to this problem seems natural since it concerns the systematic study of a theory with symmetry in order to obtain out of it the reduced effective theory. This has already been applied within the Kaluza-Klein framework where the group action is applied on a finite dimensional space.

In the present paper, in order to treat anomalies, we have extented the above concept to infinite dimensional objects. We have formulated the stratification effect the group action is imposing on the space of connections and we have pointed out its importance for a systematic treatment of anomalies. In the general case we have shown how to deal with the principal stratum. The role singular and exceptional strata are playing is at present an open question and it is still under investigation. Within the framework of G-theory, anomalies can be considered as obstructions of the quantization procedure which we meet in the process of reduction towards an effective theory. This allows the precise characterization of anomalies in terms of symmetry properties of the propagator functional before and after the fermionic degrees of freedom have been integrated out. Starting with a strictly invariant (as explained in the text) propagator functional, we meet anomalies if after the fermionic integration the propagator functional ceases to be strictly invariant, even if it still remains invariant in a more general sense. This point of view includes two major advantages. Firstly the connection of anomalies with non-trivial topological effects is very direct and plausible, and secondly a treatment of all possible anomalies on the same footing becomes possible. It therefore contributes considerably to our understanding of anomalies as topological effects of symmetries.

The above treatment and most considered effects have been demonstrated on a concrete physical model, the Aharonov-Bohm effect. The related one-dimensional field theory had infinite dimensional structures which were nontrivial but simple enough that every step could be done by an explicit calculation.

It gave an example for a gauge group with a topological structure depending on the topology of the space-time manifold, the other dependence being the topology of the structure group of the principle bundle of the gauge theory.

The stratification of the space of connections $\mathcal{A}$ under the action of the gauge group yields only one stratum due to the abelian nature of the structure group.

Although $\mathcal{A}$ and the gauge group were infinite dimensional objects, their quotient was infinite dimensional and hence easily tractable. The situation was similar to the case of the moduli spaces which are always finite dimensional: the flat connections are indeed trivially selfdual since $0 = F = F^*$. But in our case the moduli space was the complete configuration space, not a subspace.

The twist (Gribov ambiguity) of the infinite dimensional bundle $\mathcal{A}$, the principal orbit bundle, could be visualized by comparing it with the universal covering bundle $\mathbb{R}$ over S^1. In this step the language of principal fibre bundles became unavoidable.

The quantization of the fermionic degrees of freedom could be done explicitly with the ζ-function regularization. The group actions of the gauge group and other classical symmetry groups could be studied and hence the behaviour of the determinant line bundle in the reduction process by looking at our determinant as a section. Finally, the well-known parity anomaly in odd dimensions could be studied in this framework. The Aharonov-Bohm situation served as a model to study the above structures.

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REFERENCES

[1] S. L. ADLER: *Axial-vector vertex in spinor electrodynamics*, Phys. Rev. 177, 2426, 1969. J. S. BELL, R. JACKIW: *A PCAC puzzle: $\pi^0 \to \gamma\gamma$ in the σ-model*, Nuovo Cimento A60, 47, 1969. W. A. BARDEEN: *Anomalous Ward identities in spinor field theories*, Phys. Rev. 184, 1848, 1969.

[2] L. ALVAREZ-GAUMÉ, P. GINSPARG: *The topological meaning of non-Abelian anomalies*, Nucl. Phys. B243, 449, 1984. M. F. ATIYAH, L. M. SINGER: *Dirac operators coupled to vector potentials*, Proc. Nat. Acad. Sci. USA 81, 2597, 1984.

[3] I. M. SINGER: *Some remarks on the Gribov ambiguity*, Comm. Math. Phys. 60, 7, 1978. P. K. MITTER, C. M. VIALLET: *On the bundle of connections and the gauge orbit manifold in Yang-Mills theories*, Comm. Math. Phys. 79, 457, 1981. T. H. PARKER: *Gauge theories on four-dimensional Riemannian manifolds*, Comm. Math. Phys. 85, 563, 1982. T. P. KILLINGBACK, E. G. REES: *Topology of gauge theories on compact four-manifolds*, Class. Quantum Gravity 4, 357, 1987.

[4] R. CATENACCI, G. P. PIROLA, M. MARTELLINI, C. REINA: *Group actions and anomalies in gauge theories*, Phys. Lett. B 172, 223, 1986.

[5] A. HEIL, N. A. PAPADOPOULOS, B. REIFENHÄUSER, F. SCHECK: *G-theories and the geometric compactification scheme*, Nucl. Phys. B 293, 445, 1987.

[6] A. TRAUTMAN in Lect. Notes in Physics 129, ed. J. Harnad and S. Shnider, Springer, 1980. L. PERCACCI, *Geometry of nonlinear field theories*, World Scientific, 1986. L. BONORA, P. COTTA-RAMUSINO, M. RINALDI, J. STASHEFF: *The evaluation map in field theory, sigma models and strings I*, Comm. Math. Phys. 112, 237, 1987. M. C. ABBATI, R. CIRELLI, A. MANIA, P. W. MICHOR: *Smoothness of the action of the gauge transformation group on connections*, J. Math. Phys. 27, 2469, 1986.

[7] G. E. BREDON, *Introduction to compact transformation groups*, Academic Press, New York, 1972. K. JÄNICH: *Differenzierbare G-Mannigfaltigkeiten*, Lecture Notes in Mathematics 59, Springer, 1968.

[8] W. KONDRACKI, P. SADOWSKI: *Geometric structure on the orbit space of gauge connections*, Journ. Geom. Phys. 3, 421, 1986.

[9] D. J. GROSS, J. JACKIW: *Effect of anomalies on quasi-renormalizable theories*, Phys. Rev. D6, 477, 1972. L. D. FADDEEV, S. L. SHATASHVILI: *Algebraic and Hamiltonian methods in the theory of non-Abelian anomalies*, Theor. Math. Phys. 60, 770, 1985.

[10] A. HEIL, A. KERSCH, B. REIFENHÄUSER, H. VOGEL: *Application of topological ideas to the Aharonov-Bohm effect*, Europ. Journ. of Phys. 9, Nr. 3, 1988.

[11] D. BLEECKER, *Gauge theory and variational principles*, Addison-Wesley, 1981.

[12] E. WITTEN: *An $SU(2)$ anomaly*, Phys. Lett. B117, 324, 1982.

[13] D. HUSEMOLLER: *Fibre bundles*, 2^{nd} edition, Springer, 1966.

[14] S. KOBAYASHI, K. NOMIZU, *Fundations of Differential Geometry I*, John Wiley, 1963.

[15] I. M. SINGER, Seminaire Elie Cartan: *Family of Dirac operators with application to physics*, Asterisque 323, 1985.

[16] W. MAGNUS, F. OBERHETTINGER, R. P. SONI: *Formulas and theorems for the special functions of mathematical physics*, Springer, 1966.

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